Correction Model Midterm Exam LA2, March 1, 2021

1. $A = QR$ with Q orthonormal columns, R square upper triangular and positive diagonal elements.

a) $A^T A = (QR)^T QR = R^T Q^T Q R = R^T R$

Obviously R is nonsingular, R^T nonsingular, so also $R^T R$ nonsingular and

$$(A^T A)^{-1} = (R^T R)^{-1} = R^{-1} (R^T)^{-1}$$
$$= R^{-1} (R^{-1})^T$$

b) $A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

Note that A has linearly independent columns, call them a_1, a_2. I apply GS to get an orthonormal basis of span $(a_1\ a_2$.

Define $q_1 := \frac{1}{\|a_1\|} \cdot a_1$. Then

$$q_1 = \frac{1}{\sqrt{2}} a_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}. \text{ Note: } a_1 = \sqrt{2}\, q_1$$

Project a_2 onto span(q_1) to get

$$p_1 = \langle a_2, q_1 \rangle \cdot q_1 = (a_2^T q_1) \cdot q_1$$
$$= (1\ 0\ 1\ 0) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

Compute $q_2 := \dfrac{a_2 - p_1}{\|a_2 - p_1\|}$

$$a_2 - p_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$$

$$\|a_2 - p_1\| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{3}{2}}$$

So $q_2 = \sqrt{\frac{2}{3}}\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$

Note that $a_2 - p_1 = \sqrt{\frac{3}{2}}\, q_2$ so

$$a_2 = p_1 + \sqrt{\tfrac{3}{2}}\, q_2$$
$$= \frac{1}{\sqrt{2}} q_1 + \frac{\sqrt{3}}{\sqrt{2}} q_2$$

This yields $(a_1 \;\; a_2) = (q_1 \;\; q_2)\begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{3/2} \end{pmatrix}$,

so:

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{6} \\ 0 & \sqrt{2/3} \\ 0 & 0 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{3/2} \end{pmatrix}}_{R}$$

c) $A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

The least squares solution to this equation is the unique solution of $A^T A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\Leftrightarrow \quad R^T Q^T Q R \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = R^T Q^T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \quad R \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Q^T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now, $Q^T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$R \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \quad \Leftrightarrow$$

$$\begin{pmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{3/2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$x_2 = \sqrt{\frac{2}{3}} \cdot \sqrt{\frac{1}{6}} = \sqrt{\frac{2}{18}} = \frac{\sqrt{2}}{\sqrt{2}\sqrt{9}} = \frac{1}{3}$$

$$\sqrt{2}\, x_1 + \frac{1}{\sqrt{2}}\ \frac{1}{3} = \frac{1}{\sqrt{2}}$$

$$\sqrt{2}\, x_1 = \frac{1}{\sqrt{2}}\left(1 - \frac{1}{3}\right) = \frac{2}{3\sqrt{2}}$$

$$x_1 = \frac{1}{\sqrt{2}}\ \frac{2}{3\sqrt{2}} = \frac{1}{3}$$

Other method: least squares solution is the unique solution of $A^TAx = A^Tb$

$$A^TA = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$(A^TA)^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$A^Tb = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow x = (A^TA)^{-1}A^Tb$$

$$= \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

Same of course.

2. a) We should check the three axioms for inner product:

(i) $\langle f, f\rangle \geq 0 \quad \forall f$

yes: $\langle f, f\rangle = \int_0^1 f(x)^2 dx + 2f(0)^2 + f(1)^2 \geq 0$

$\langle f, f\rangle = 0 \Rightarrow \int_0^1 f(x)^2 dx = 0 \Leftrightarrow f(x)=0 \ \forall x \in [0,1]$

Conversely, $f=0 \Rightarrow \langle f, f\rangle = 0$ as well.

(ii) $\langle f, g\rangle = \langle g, f\rangle$ is obvious.

(iii) $\langle \alpha f+\beta g, h\rangle =$

$$\int_0^1 (\alpha f+\beta g)(x)\, h(x)\, dx + 2(\alpha f+\beta g)(0)\, h(0) + (\alpha f+\beta g)(1)\, h(1)$$

$$= \alpha\left(\int_0^1 f(x) h(x) dx + 2f(0)h(0) + f(1)h(1)\right)$$
$$\beta\left(\int_0^1 g(x) h(x) dx + 2g(0)h(0) + g(1)h(1)\right)$$

$$= \alpha\langle f, h\rangle + \beta\langle g, h\rangle$$

b) $\|f\|^2 = \int_0^1 1.1\, dx + 2.1.1 + 1.1 = 1+2+1 = 4$

$\Rightarrow \|f\| = 2$

$\|g\|^2 = \int_0^1 x^2 dx + 1.1 = \left[\frac{1}{3}x^3\right]_0^1 + 1 = \frac{4}{3}$

$\|g\| = \frac{2}{\sqrt{3}}$

c.) $\cos\Theta = \dfrac{\langle f, g\rangle}{\|f\|\,\|g\|}$.

$$\langle f, g\rangle = \int_0^1 x\,dx + 1.1 = \left[\tfrac{1}{2}x^2\right]_0^1 + 1 = \frac{3}{2}$$

$$\Rightarrow \cos\Theta = \frac{1}{2\cdot\frac{2}{\sqrt{3}}}\cdot\frac{3}{2} = \frac{3\sqrt{3}}{8}$$

d) $S = \operatorname{span}(1, x)$. We make an orthonormal basis $\{u_1(x), u_2(x)\}$ using Gram–Schmidt.

$$u_1(x) := \frac{1}{\|1\|}\cdot 1 = \frac{1}{2}$$

Let $p_1(x)$ be the projection of x onto $\operatorname{span}(\frac{1}{2})$

$$p_1(x) = \langle x, \tfrac{1}{2}\rangle\cdot\tfrac{1}{2}$$

$$= \int_0^1 \tfrac{1}{2}x\,dx + 1.\tfrac{1}{2} = \left[\tfrac{1}{4}x^2\right]_0^1 + \tfrac{1}{2}$$

$$= 3/4$$

Define then $u_2(x) = \dfrac{x - 3/4}{\|x - 3/4\|}$

$$\|x - 3/4\|^2 = \int_0^1 \left(x - \tfrac{3}{4}\right)^2 dx + 2.\left(-3/4\right)^2 + \left(\tfrac{1}{4}\right)^2$$

$$= \left[\tfrac{1}{3}\left(x - \tfrac{3}{4}\right)^3\right]_0^1 + 2.\frac{9}{16} + \frac{1}{16}$$

$$= \frac{1}{3}\cdot\left(\frac{1}{4}\right)^3 - \frac{1}{3}\left(-\frac{3}{4}\right)^3 + \frac{18}{16} + \frac{1}{16}$$

$$= \frac{1}{3}\,\frac{1}{64} + \frac{1}{3}\,\frac{27}{64} + \frac{72}{64} + \frac{4}{64}$$

$$= \frac{1}{3}\,\frac{1}{64} + \frac{9}{64} + \frac{72}{64} + \frac{4}{64}$$

$$= \frac{1}{64}\left(85 + \frac{1}{3}\right) = \frac{1}{64}\,\frac{256}{3} = \frac{4}{3}$$

$$\Rightarrow \left\| x - \frac{3}{4} \right\| = \frac{2}{\sqrt{3}}$$

$$u_2(x) = \frac{1}{2}\sqrt{3}\left(x - \frac{3}{4}\right)$$

3) $A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

a) $\det(A - sI) = \det \begin{pmatrix} 2-s & 1 & 0 \\ -1 & 1-s & 1 \\ 1 & 0 & -s \end{pmatrix}$

$= (2-s)(1-s)(-s) + (-s) + 1$

$= -s^3 + 3s^2 - 3s + 1$

b) $-s^3 + 3s^2 - 3s + 1 = (1-s)^3$

Hence there is one eigenvalue : $\lambda = 1$.

c) Solve $\begin{pmatrix} 2 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$2x + y = x \quad \Rightarrow \quad x = -y$

$-x + y + z = y$

$x = z$

So: any eigenvector is of the form

$\lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \qquad \lambda \neq 0.$

d) Not diagonalisable because A does NOT have 3 linearly independent eigenvectors.

4. A is unitary, i.e. $AA^H = A^H A = I$.

a) Let λ be an eigenvalue and $Ax = \lambda x$, $x \neq 0$.
Then $(Ax)^H Ax = x^H A^H A x = \|x\|^2$.
Also $(Ax)^H Ax = (\lambda x)^H \lambda x = |\lambda|^2 \|x\|^2$.
Hence $|\lambda|^2 = 1$ so $|\lambda| = 1$.

b) Assume $|\lambda| = 1$. Then $\lambda \neq 0$. Also $|\lambda|^2 = 1$
so $\bar{\lambda}\lambda = 1$. This yields $\bar{\lambda} = \frac{1}{\lambda}$

c) Let $\lambda_1 \neq \lambda_2$ eigenvalues and x_1, x_2 corresponding eigenvectors. Need to show $x_1^H x_2 = 0$.

Note that $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$.
Thus $(Ax_1)^H A x_2 = x_1^H A^H A x_2 = x_1^H x_2$.
Also $(Ax_1)^H A x_2 = (\lambda_1 x_1)^H \lambda_2 x_2$
$= \bar{\lambda}_1 \lambda_2 x_1^H x_2 = \frac{\lambda_2}{\lambda_1} x_1^H x_2$.
Hence $x_1^H x_2 = \frac{\lambda_2}{\lambda_1} x_1^H x_2$. Since $\frac{\lambda_2}{\lambda_1} \neq 1$
we must have $x_1^H x_2 = 0$

d) Yes: $A^H A = A A^H$ so A is normal,
so unitarily diagonalisable.